ON CAPUTO AND RIEMANN-LIOUVILLE DERIVATIVES OF FRACTIONAL ORDER

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ABSTRACT: *Fractional derivative is a field of mathematical analysis which deals with derivatives of arbitrary order. Riemann-Liouville and Caputo introduced different formulas for computing the fractional derivatives. Indeed, Caputo derivative does not coincide with the Riemann-Liouville derivative. This paper investigates Caputo and Riemann-Liouville derivatives of fractional order. Relevant information and known properties on Caputo and Riemann-Liouville derivatives of fractional order were considered to establish the results of this study. These results are proved by direct method. This paper provides some results about the Caputo and Riemann-Liouville derivatives of fractional orders. It also examines the Caputo and Riemann-Liouville derivatives of some series. Moreover, it provides necessary and sufficient conditions so that the Caputo and Riemann-Liouville derivatives of fractional order coincide.*

Keywords: Fractional Derivatives, Caputo Derivatives, Riemann-Liouville Derivatives, Theoretical Research, Philippines

1. INTRODUCTION

Fractional calculus is the study of generalized orders of differentiation and integration (together referred to as differ-integration) beyond integer orders to real numbers, and complex numbers. It has its origin way back 1695 when L'Hospital raised question whether the meaning of derivative of order n would still be valid when n is not an integer. The advantage of using fractional derivative versus the integer derivative is that the integer derivative is local in nature, where as the fractional derivative is global in nature. Riemann-Liouville formulated the formula for computing the fractional derivatives. However, in 1967, Caputo introduced another way for computing the fractional derivatives. Indeed, Caputo derivative does not coincide with the classical derivative while Riemann-Liouville derivative is in-line with the classical derivative. With this, the existence of the derivative of a function in Caputo sense is fewer than those in Riemann-Liouville sense. In this paper, the researchers are motivated to examine functions in which Caputo and Riemann-Liouville derivatives of fractional order coincide.

One of the applications of fractional calculus is the fractional order proportional-integral-derivative controller (PID controller) which is an automatic controller widely used in the industrial control systems nowadays. PID controller gives a more effective way to enhance the control performance of the system. Moreover, in concrete terms, it automatically applies accurate and responsive correction to a control function. For instance, the cruise control on a car, where external influences such as hills (gradients) would decrease speed. The PID algorithm restores from current speed to the desired speed in an optimal way, without delay or overshoot, by controlling the power output of the vehicle's engine.

2. MATERIAL AND METHODS

This study is theoretical research and hence the results are the proofs and theorems generated. Relevant information and known properties on Caputo and Riemann-Liouville derivatives of fractional order are considered from published materials such as books, journals and online sources. Furthermore, preliminary concepts are presented, which are useful to prove the results of this study.

3. PRELIMINARIES

In this section, basic definitions of Caputo and Riemann-Liouville of fractional derivatives and fractional integrals are presented. Additionally, some known properties are also discussed, as they play an important role in proving the subsequent results. Furthermore, examples are provided to illustrate the concepts of Riemann-Liouville and Caputo derivatives of fractional order.

Definition 3.1. The *fractional integral* (or the *Riemann-Liouville integral*) $D^{-\alpha}$ with fractional order $\alpha \in \mathbb{R}^+$ of function f is defined as

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)ds.$$

In this context, the integration runs from 0 to x.

Definition 3.2. The *Riemann-Liouville derivative of fractional order* α of function f is given by

$$D_{RL}^{\alpha} f(x) = \frac{d^n}{dt^n} D^{-(n-\alpha)} f(x)$$
$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-s)^{n-\alpha-1} f(s)ds$$

where $n-1 \leq \alpha < n$, where $n \in \mathbb{Z}^+$.

Example 3.3. Consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5$. Let $n = 1$. Then the Riemann-Liouville derivative of fractional order $\alpha = \frac{1}{2}$ is

$$D_{RL}^{\frac{1}{2}} f(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dx} \int_0^x (x-s)^{-\frac{1}{2}} (5)\, ds = -\frac{5x^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}.$$

Since $\Gamma\left(\frac{1}{2}\right) = \pi$, $D_{RL}^{\frac{1}{2}} f(x) = -\frac{5}{\pi\sqrt{x}}$.

Definition 3.4. The *Caputo derivative of fractional order* α of function f is defined as

$$D_C^{\alpha} f(x) = D^{-(n-\alpha)} \frac{d^n}{dx^n} f(x)$$
$$= \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} \frac{d^n}{ds^n} f(s)\, ds$$

where $n-1 < \alpha < n$, where $n \in \mathbb{Z}^+$.

Example 3.5. Consider a function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2$ for all $x\in\mathbb{R}$. Let $n=1$. Then the Caputo derivative of fractional order $\frac{1}{2}$ is

$$\begin{aligned} D_C^{\frac{1}{2}}f(x) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-s)^{-\frac{1}{2}}\frac{d}{ds}s^2ds \\ &= \frac{2}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-s)^{-\frac{1}{2}}s\,ds \\ &= \frac{2}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{4}{3}x^{\frac{3}{2}}\right) \\ &= \frac{8x^{\frac{3}{2}}}{3\pi}. \end{aligned}$$

Caputo derivative does not coincide with the classical derivative (Li & Deng, 2007), say, for $\alpha\in(m-1,m),\ m\in\mathbb{Z}^+$,

$$\lim_{\alpha\to(m-1)^+} D_C^{\alpha}f(x)=f^{(m-1)}(x)-f^{(m-1)}(0)$$

and

$$\lim_{\alpha\to(m)^+} D_C^{\alpha}f(x)=f^{(m)}(x)$$

where $f^{(m)}(x)=\frac{d^m}{dx^m}f(x)$, while RL derivative is in-line with the classical derivative, say, for $\alpha\in(m-1,m),\ m\in\mathbb{Z}^+$,

$\lim_{\alpha\to(m-1)^+} D_{RL}^{\alpha}f(x)=f^{(m-1)}(x)$

and

$\lim_{\alpha\to(m)^+} D_{RL}^{\alpha}f(x)=f^{(m)}(x)$.

With this, the existence of the derivative of a function in Caputo sense is fewer than those in RL sense.

The following properties of Caputo and RL derivatives are taken from (Li & Deng, 2007; Li, et al., 2009; Li, et al., 2011; Podlubny, 1999).

1. If $\alpha>0$, then for any positive integer n,
   $\frac{d^n}{dx^n}D_{RL}^{\alpha}f(x)=D_{RL}^{n+\alpha}f(x)$;
   $$\frac{d^n}{dx^n}D_{RL}^{-\alpha}f(x)=D_{RL}^{n-\alpha}f(x) \quad \text{if } n-\alpha>0, \text{and}$$
   $\frac{d^n}{dx^n}D_{RL}^{-\alpha}f(x)=D^{n-\alpha}f(x)$ if $n-\alpha<0$.
2. For all $\alpha>0$, $(D_{RL}^{\alpha}\cdot D^{-\alpha})f(x)=f(x)$. More generally,
   $(D_{RL}^{\alpha}\cdot D^{-\beta})f(x)=D_{RL}^{\alpha-\beta}f(x)$ for all $\beta>0$.
   If $\alpha<\beta$, then $D_{RL}^{\alpha-\beta}f(x)=D^{\alpha-\beta}f(x)$.
3. $(D^{-n}\cdot D_{RL}^{n})f(x)=f(x)-\sum_{k=0}^{n-1}\left(\frac{x^k}{k!}\right)f^{(k)}(0)$.
4. $D_{RL}^{\alpha}(c)=\frac{cx^{-\alpha}}{\Gamma(1-\alpha)}$, where $\alpha>0$ and c is an arbitrary constant.
5. $D_C^{\alpha}f(x)=D_{RL}^{\alpha}\left(f(x)-\sum_{k=0}^{n-1}\left(\frac{x^k}{k!}\right)f^{(k)}(0)\right)$, where $n-1<\alpha<n\in\mathbb{Z}^+$.
6. $f(x)\in C^1[0,x'], x'>0,\ \alpha_i\in(0,1)(i=1,2)$ (the trivial case $\alpha_i=0$ or 1 is simple and removed here), and $\alpha_1+\alpha_2\in(0,1]$, then
   $$D_C^{\alpha_1+\alpha_2}f(x)=D_{RL}^{\alpha_1+\alpha_2}f(x)-\frac{x^{-\alpha_1-\alpha_2}}{\Gamma(1-\alpha_1-\alpha_2)}f(0).$$
7. If $\alpha\in(n-1,n), n\in\mathbb{Z}^+$, and $f^{(k)}(0)\geq 0$, $(k=0,1,2,\ldots,n-1)$, then $D_C^{\alpha}\geq D_{RL}^{\alpha}$.

Note that if $\alpha_1+\alpha_2=1$, then $\frac{x^{-\alpha_1-\alpha_2}}{\Gamma(1-\alpha_1-\alpha_2)}f(0)$ in Property 6 becomes zero since $\Gamma(0)=\infty$.

4. RESULTS AND DISCUSSION

In this study, the investigation of Caputo and Riemann-Liouville derivatives has yielded valuable insights into the behavior of certain mathematical series. Moreover, the necessary and sufficient conditions governing the convergence of Caputo and Riemann-Liouville derivate are examined. The following presents the results of this study.

Theorem 4.1. *If $f(x)=ax^n$ where $a\in\mathbb{R}$ and $n\in\mathbb{Z}^+$, then* $\sum_{k=0}^{n-1}\left(\frac{t^k}{k!}\right)f^{(k)}(0)=0$.

Proof: Note that $f^{(k)}(x)=\frac{an!}{(n-k)!}x^{n-k}$ for any $k=0,1,2,\ldots,n-1$. It follows that $f^{(k)}(0)=0$ for all $k=0,1,2,\ldots,n-1$. Hence $\sum_{k=0}^{n-1}\left(\frac{x^k}{k!}\right)f^{(k)}(0)=0$. ■

Theorem 4.2. *If $f(x)=\sum_{i=n}^{m}a_ix^i$ where $a\in\mathbb{R}$ and $n,m\in\mathbb{Z}^+$, then* $\sum_{k=0}^{n-1}\left(\frac{x^k}{k!}\right)f^{(k)}(0)=0$.

Proof: The proof is similar to Theorem 4.1. ■

Theorem 4.3. *If f is a function defined by $f(x)=\sum_{i=n}^{m}a_ix^i$ where $a\in\mathbb{R}$ and $n,m\in\mathbb{Z}^+$, then $D_C^{\alpha}f=D_{RL}^{\alpha}f$.*

Proof: By Theorem 4.2, $\sum_{k=0}^{n-1}\left(\frac{x^k}{k!}\right)f^{(k)}(0)=0$. Thus, $D_C^{\alpha}f=D_{RL}^{\alpha}f$. ■

Theorem 4.4. *Let f be a differentiable function and has bounded derivative. Then f is continuously differentiable on $[0,x']$ and $D_{RL}^{\alpha}(1)=0$ for $\alpha\in(0,1)$, if and only if $D_C^{\alpha}f=D_{RL}^{\alpha}f$.*

Proof: Note that $D_{RL}^{\alpha}(1)=\frac{x^{-\alpha}}{\Gamma(1-\alpha)}=0$. Thus, we obtain

$$D_C^{\alpha}f(x)=D_{RL}^{\alpha}f(x)-\frac{x^{-\alpha}}{\Gamma(1-\alpha)}f(0)=D_{RL}^{\alpha}f(x).$$

Conversely, suppose $D_C^{\alpha}f=D_{RL}^{\alpha}f$. Then, for $\alpha\in(0,1)$, we have

$$\begin{aligned} D_{RL}^{\alpha}(1) &= D_C^{\alpha}(1) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-s)^{n-\alpha-1}\frac{d^n(1)}{ds^n}\,ds=0. \end{aligned}$$

Now, since every differentiable function that has bounded derivative is uniformly continuous and every uniformly continuous function is continuous, it follows that f is continuously differentiable function on $[0,x']$.

The following definition will be used in the next result.

Definition 4.5. If K is a field, then we can define the field of *Puiseux series* with coefficients in K informally as the set of expressions of the form

$$f=\sum_{k=k_0}^{+\infty} c_k T^{\frac{k}{n}}$$

where n is a positive integer and k_0 is an arbitrary integer.

Theorem 4.6. *If f is a Puiseux series defined by $f(x)=\sum_{m=m_0}^{+\infty} c_m x^{\frac{m}{n}}$, then*

$$D_C^{\alpha} f = D_{RL}^{\alpha} f \text{ where } \left\lceil\frac{m}{n}\right\rceil - 1 < \alpha < \left\lceil\frac{m}{n}\right\rceil.$$

Proof: We first claim that $\sum_{k=0}^{\left\lceil\frac{m}{n}\right\rceil-1}\left(\frac{x^k}{k!}\right) f^{(k)}(0)=0$. Note that

$$f(x)=c_{m_0} x^{\frac{m_0}{n}} + c_{m_1} x^{\frac{m_1}{n}} + \cdots$$

implies

$$f^{(k)}(x)=\frac{c_{m_0}\left(\frac{m_0}{n}\right)!}{\left(\frac{m_0}{n}-k\right)!} x^{\frac{m_0}{n}-k} + \frac{c_{m_1}\left(\frac{m_1}{n}\right)!}{\left(\frac{m_1}{n}-k\right)!} x^{\frac{m_1}{n}-k} + \cdots$$

$$=\sum_{m=m_0}^{+\infty} \frac{c_m\left(\frac{m}{n}\right)!}{\left(\frac{m}{n}-k\right)!} x^{\frac{m}{n}-k}.$$

Observe that $f^{(k)}(0)=0$ for all $k=0,1,2,\dots,\left\lceil\frac{m}{n}\right\rceil-1$. Thus

$$\sum_{k=0}^{\left\lceil\frac{m}{n}\right\rceil-1}\left(\frac{x^k}{k!}\right) f^{(k)}(0)=0.$$

Therefore, for $\left\lceil\frac{m}{n}\right\rceil-1<\alpha<\left\lceil\frac{m}{n}\right\rceil$, we have

$$D_C^{\alpha} f(x) = D_{RL}^{\alpha} f(x). \qquad \blacksquare$$

5. CONCLUSION

In general, the Caputo and Riemann-Liouville derivatives do not coincide. Hence this paper showed some conditions so that the Caputo and Riemann-Liouville derivatives coincide. Moreover, for any differentiable function with bounded derivative, the two derivatives are equal if and only if the function is continuously differentiable on a closed interval and the Riemann-Liouville derivative of 1 of fractional order $\alpha \in (0,1)$ is 0. The results of this study are all theories, hence an investigation of concrete application of these results is recommended.

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